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A KERR-VAIDYA RADIATING METRIC

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "A KERR-VAIDYA RADIATING METRIC" submitted by MARTIN MURENBEELD, B.Sc. (Honours) in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The object of this thesis is to present a possible exterior metric for a rotating, radiating spherical body. The exterior geometry of a non-rotating radiating spherical body is determined by Vaidya's "radiating Schwarzschild metric", whereas it is generally accepted that a rotating Schwarzschild body leads to the Kerr metric. However, no one has as yet presented a solution which describes both rotation and radiation.

Although the Vaidya and Kerr metrics are discussed, we are primarily concerned with obtaining a Kerr analogue to the Vaidya metric. The results obtained, with the principle limitation that the angular momentum per unit mass remains constant, appear to be plausible generalizations of those due to Vaidya. In particular, we have found that the system loses angular momentum at a rate equal to $-m'a$, where $-m'$ is the mass radiated per unit time. The appropriate null geodesics have been computed and the results indicate that the velocity of the photons has acquired a tangential component.

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CHAPTER I

INTRODUCTION

In 1905 Albert Einstein [1] formulated what is now known as the Special Theory of Relativity. This theory was a radical departure from classical physics but it has been experimentally vindicated to the point where it has become an integral part of modern physics. The special theory as reformulated by Minkowski [2] was one of the first steps towards taking a geometrical view of physics. Indeed, it was the ideas of Minkowski which laid the foundation for Einstein's [3] 1915 theory, commonly referred to as the General Theory of Relativity. This theory was the first attempt to explain certain physical phenomena in terms of purely geometrical entities. Since then many attempts have been made to completely "geometrize" physics but no generalization of the 1915 theory has proven to be without defect. Consequently the General Theory of Relativity stands today as the best theory of gravitation, and it is with this theory that this dissertation is concerned.

The general theory stipulates that the events in the real world can be considered as points in a four-dimensional Riemannian space of signature ± 2. The exact form of the metric tensor for the space is determined from the field equations

$$R_{ij} - \frac{1}{2} g_{ij}R = - 8\pi T_{ij} \tag{1.1}$$

where R_{ij} is the Ricci tensor, R the curvature invariant $g_{ij}R^{ij}$, and T_{ij} is the energy momentum tensor of the space under consideration. The units have been chosen so that the speed of light and the Newtonian gravitational constant have unit value. The left hand side of (1.1) contains purely geometrical quantities while the physical aspects of space time enter by way of the tensor T_{ij}. In a local Minkowski reference frame T_{ij} is given by (assuming no heat flow and comoving)

$$T_{ij} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & t_{xx} & t_{xy} & t_{xz} \\ 0 & t_{yx} & t_{yy} & t_{yz} \\ 0 & t_{zx} & t_{zy} & t_{zz} \end{pmatrix} \tag{1.2}$$

where t_{ij} is the three dimensional stress tensor and q the matter density. In empty space T_{ij} is identically zero. In view of this fact it follows that R equals zero. Hence, field equation (1.1) reduces to

$$R_{ij} = 0. \tag{1.3}$$

Within a few months after the appearance of the general theory

Karl Schwarzschild [4] showed that the metric tensor corresponding to the line element

$$ds^2 = -(1-\frac{2m}{r})^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 + (1-\frac{2m}{r})dt^2 \tag{1.4}$$

was a solution to (1.3). This metric, by now the most well known solution to Einstein's field equations, describes the geometry exterior to a static, non-radiating, spherical body of constant mass m.

Later that same year H. Reisnner and in 1918, G. Nordström [5] obtained a generalization of the Schwarzschild solution. Their line element took the form

$$\begin{aligned} ds^2 = &-(1-\frac{2m}{r}+\frac{4\pi\varepsilon^2}{r^2})^{-1}dr^2 \\ &-r^2(d\theta^2+\sin^2\theta d\phi^2) \\ &+(1-\frac{2m}{r}+\frac{4\pi\varepsilon^2}{r^2})dt^2 \end{aligned} \tag{1.5}$$

which describes the exterior geometry determined by a Schwarzschild particle with charge ε .

The Schwarzschild solution was particularly valuable in that it provided experimentalists with a tool to check the validity of the general theory. It was basic to the "three crucial tests",

which concerned the perihelic shift in Mercury's orbit, the bending of light rays as they passed near a massive body and the gravitational red shift. The net result of these tests was a significant triumph for relativity theory, particularly in that the red shift and light bending phenomena were observed only <u>after</u> they had been predicted by the general theory. Electrical effects predicted by the Reisnner-Nordström metric are in general of a magnitude which does not lend itself to experimental detection.

An interesting problem considered by P.C. Vaidya [6] is that of a radiating Schwarzschild particle. The sun, for example, is an excellent approximation to a Schwarzschild body. However, the space surrounding the sun is evidently filled with radiation, therefore we must assume that T_{ij} is not identically zero in that region. Vaidya argued that T_{ij} should have the form

$$T_{ij} = qv_i v_j \tag{1.6}$$

where q is the energy density of the radiation and v_i is the propogation null vector. From this it follows that the field equations for the radiation zone may be written as

$$R_{ij} = - 8\pi T_{ij} \ . \tag{1.7}$$

Vaidya's solution to (1.6), (1.7) gave rise to the metric

$$ds^2 = -(1-\frac{2m}{r})^{-1}dr^2 - r^2(d\theta^2+\sin^2\theta d\phi^2)$$
$$+(\frac{\dot{m}}{f})^2(1-\frac{2m}{r})dt^2 \tag{1.8}$$

where $m = m(r,t)$ and $f(m)$ is an arbitrary function subject to the relation $m'(1-\frac{2m}{r}) = f(m)$. The prime and the dot indicate differentiation with respect to r and t respectively. A more detailed discussion of this solution is given in Chapter II.

Another variation of the Schwarzschild solution should arise in the case in which a spherically symmetric body is rotating in an empty universe. In 1963, R.P. Kerr [7] discovered a metric apparently compatible with the above case. This metric can be put in the form

$$\begin{aligned} ds^2 = &- \rho^2[\frac{dr^2}{r^2-2mr+a^2} + d\theta^2] \\ &- [r^2+a^2+ \frac{2mra^2\sin^2\theta}{\rho^2}] \sin^2\theta d\phi^2 \\ &- \frac{4mar\sin^2\theta}{\rho^2} d\phi dt \\ &+ (1- \frac{2mr}{\rho^2})dt^2 \end{aligned} \tag{1.9}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$.

Weak field considerations suggest that if m is to be the usual Schwarzschild mass then ma should be identified with

the angular momentum of the body. The Kerr metric (1.9) was later generalized by E.T. Newman et al. [8] to the case where m is charged.

In the discussion in the following chapters we consider another variation of the Schwarzschild theme and present one form of a "Kerr-Vaidya" metric. That is, we apply the Vaidya technique to the Kerr metric, the results of which bear relevance to the problem of a rotating, radiating spherical body.

CHAPTER II

THE PROBLEM OF VAIDYA

As mentioned in the previous chapter an interesting problem is that of the gravitational field of a radiating Schwarzschild body. Earlier considerations [9] of this problem were for the most part unsuccessful as they were attempts at generalizing the Schwarzschild solution to non static masses. Hence it was not until 1950, when P.C. Vaidya first exhibited a physically appealing solution, that significant result was obtained. The first consideration was in fact to display a meaningful energy momentum tensor and the following is based on Vaidya's work.

The electromagnetic energy momentum tensor is given by [10]

$$T^{ij} = -g^{jk}F^{im}F_{km} + \frac{1}{4}\, g^{ij}F^{km}F_{km} \qquad (2.1)$$

where F_{ij} is the electromagnetic field tensor which when expressed in Minkowski coordinates takes the form:

$$F_{ij} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & H_z & -H_y \\ E_y & -H_z & 0 & H_x \\ E_z & H_y & -H_x & 0 \end{pmatrix} \tag{2.2}$$

The coordinate notation $x^1, x^2, x^3, x^4 = t, x, y, z$ is used, and the summation is from 1 to 4. In terms of the above coordinates we obtain the typical examples

$$\begin{aligned} T^1_1 &= \frac{1}{2}\,(E_x^2+E_y^2+E_z^2+H_x^2+H_y^2+H_z^2) \\ T^2_2 &= \frac{1}{2}\,(E_x^2-E_y^2-E_z^2+H_x^2-H_y^2-H_z^2) \\ T^2_3 &= T^3_2 = E_xE_y + H_xH_y \\ T^2_1 &= -T^1_2 = E_yH_z - E_zH_y\ . \end{aligned} \tag{2.3}$$

If we now consider a directed flow of radiation -- meaning that a local observer at any point in the region of space under consideration will find one and only one direction in which the radiant energy is flowing, -- directed for simplicity in the x (x^2) direction and assume that the radiation is plane polarized with its electric vector parallel to the y (x^3) direction we obtain

$$\begin{aligned} E_x = E_z = H_x = H_y = 0 \\ E_y = H_z\ . \end{aligned} \tag{2.4}$$

It follows from (2.3) that

$$T^1_1 = -T^2_2 = T^2_1 = -T^1_2$$
$$= \frac{E_y^2 + H_z^2}{2} \equiv q \tag{2.5}$$

all the other T^i_j being identically zero. q is evidently the density of the radiant energy at the point in question. This result also holds for the average of incoherent unpolarized radiation.

The energy momentum tensor in any other coordinate system may be found by the tensor transformation rule

$$T^{ij} = \frac{\partial x^i}{\partial x_o^\mu} \frac{\partial x^j}{\partial x_o^\beta} T^{\mu\beta}. \tag{2.6}$$

The subscript o has been introduced to designate natural coordinates. As the radiation flows along a null geodesic we must have that

$$\begin{aligned} ds_o^2 &= (g_{\mu\beta})_o dx_o^\mu dx_o^\beta \\ &= (dx_o^1)^2 - (dx_o^2)^2 \\ &= 0 \end{aligned} \tag{2.7}$$

or,

$$dx_o^1 = dx_o^2 = d\tau. \tag{2.8}$$

Now since

$$\begin{aligned} \frac{dx^i}{d\tau} &= \frac{\partial x^i}{\partial x_o^\mu} \frac{dx_o^\mu}{d\tau} \\ &= \frac{\partial x^i}{\partial x_o^1} + \frac{\partial x^i}{\partial x_o^2} \end{aligned} \tag{2.9}$$

it follows from (2.5), (2.6) and (2.9) that

$$T^{ij} = \frac{dx^i}{d\tau}\frac{dx^j}{d\tau} q$$

or, denoting $\frac{dx^i}{d\tau}$ by v^i, we have

$$T_j^i = qv^iv_j. \tag{2.10}$$

v_i is interpreted as the propogation null vector of the radiation. Also, since $v^iv_i = 0$ we have that

$$R = 0; \tag{2.11}$$

it follows that the field equations for directed radiation are

$$R_{ij} = - 8\pi q v_i v_j . \tag{2.12}$$

In the above discussion the radiation was directed along the x-axis. In the spherically symmetric case the space part of v_i would be in the radial direction.

We will now suppose that a star of initial mass M and radius R starts to radiate at time t_o. At some later instant t_1 the radiation has reached $r = r_1$ so that the region

$$R \leq r \leq r_1, \qquad t_o \leq t \leq t_1 \tag{2.13}$$

is the zone of radiation wherein (2.12) holds. Vaidya assumed a metric of the form

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{2.14}$$

with $\lambda = \lambda(r,t)$, $\nu = \nu(r,t)$. If we now take the flow of radiation to have radial direction we will have that

$$v^3 = v^4 = 0. \tag{2.15}$$

This reduces (2.12) to three non-trivial equations to which we

must add the null condition

$$e^{\nu}(v^1)^2 - e^{\lambda}(v^2)^2 = 0. \tag{2.16}$$

The general solution to (2.12) and (2.16) gives rise to the line element

$$\begin{aligned} ds^2 = (\frac{\dot{m}}{f})^2(1-\frac{2m}{r})dt^2 - (1-\frac{2m}{r})^{-1}dr^2 \\ - r^2(d\theta^2+\sin^2\theta d\phi^2) \end{aligned} \tag{2.17}$$

where $m = m(r,t)$ and $f(m) = m'(1-\frac{2m}{r})$ is an arbitrary function.

A more revealing form of Vaidya's line element was given in 1953 [11] and may be written as

$$\begin{aligned} ds^2 = (1-\frac{2m(u)}{r})du^2 + 2dudr \\ - r^2(d\theta^2+\sin^2\theta d\phi^2) \end{aligned} \tag{2.18}$$

where r is a null coordinate and u a retarded time coordinate. A transformation to explicitly diagonalize (2.18) with $\frac{dm}{du} \neq 0$ is not known but Finkelstein [12] showed that the transformation

$$\begin{aligned} m &= \text{constant} \\ u &= t - r \\ T &= t + 2m \ln(r-2m) \end{aligned} \tag{2.19}$$

will produce the Schwarzschild metric (1.4) with coordinates (T,r,θ,ϕ). Hence line element (2.18) wherein $m(u)$ is a constant function is simply the Schwarzschild line element in "radiation coordinates".

The only non zero component of the energy momentum tensor in the (u,r,θ,ϕ) coordinate system is $T_{uu} = T_{11}$. It turns out that

$$- 8\pi T_{11} = \frac{2m'}{r^2} = - 8\pi q(v_1)^2 \tag{2.20}$$

from which it follows that

$$(v_1)^2 = \frac{- m'}{4\pi r^2 q} . \tag{2.21}$$

Since the energy density $q > 0$ we must have that $m' < 0$, indicating that the star is losing mass. If we normalize the propogation null vector v_i such that $v_1 = 1$ we have that

$$q = \frac{- m'}{4\pi r^2} . \tag{2.22}$$

Lindquist, Schwartz and Misner [13] point out however, that the energy density given in (2.22) is not the density as

measured by an observer travelling with four-velocity w^i. They define the observed energy density $\bar{q}$ as

$$\bar{q} = T_{ij} w^i w^j . \tag{2.23}$$

In natural coordinates wherein $w^i = (1,0,0,0)$ it is seen that $\bar{q}$ may be identified with the q of (1.2). If we now assume the observer to be moving radially we have

$$\begin{aligned} w^1 &= \frac{du}{d\tau} & w^2 &= \frac{dr}{d\tau} = U \\ w^3 &= \frac{d\theta}{d\tau} = 0 & w^4 &= \frac{d\phi}{d\tau} = 0 \end{aligned} \tag{2.24}$$

and we obtain

$$\begin{aligned} \bar{q} &= -\frac{1}{8\pi} w^i w^j R_{ij} \\ &= \frac{-m'}{4\pi r^2} (w^1)^2 . \end{aligned} \tag{2.25}$$

The condition

$$w^i w_i = 1 \tag{2.26}$$

implies that

$$w^1 = \frac{1}{\alpha+U} \tag{2.27}$$

where $\alpha = (U^2+1-\frac{2m}{r})^{\frac{1}{2}}$. Therefore

$$\bar{q} = q\left(\frac{1}{\alpha+U}\right)^2 . \tag{2.28}$$

If we now define luminosity as

$$L = 4\pi r^2\bar{q}, \tag{2.29}$$

the observed radiation density seen at some distance r from the star, we find that an observer at rest at infinity would calculate a total luminosity of

$$L_\infty = \lim_{\substack{r \to \infty \\ U \to o}} 4\pi r^2\bar{q} \tag{2.30}$$

$$= -m'.$$

In other words, the total luminosity is equal to the mass radiated per unit time.

CHAPTER III

THE KERR METRIC

In 1963 a new solution to the empty space field equations (1.3) appeared. R. Kerr [14], while investigating algebraically special solutions of Einstein's empty space field equations, discovered the stationary, axially symmetric vacuum metric*

$$
\begin{aligned}
ds^2 = {} & (1-\frac{2mr}{\rho^2})du^2 + 2dudr - 2a\sin^2\theta drd\phi \\
& + \frac{4mra\sin^2\theta}{\rho^2}dud\phi - \rho^2 d\theta^2 \\
& - \sin^2\theta(r^2+a^2+\frac{2mra^2\sin^2\theta}{\rho^2})d\phi^2
\end{aligned}
\tag{3.1}
$$

where $\rho^2 = r^2 + a^2\cos^2\theta$. Much was already known about static axially symmetric vacuum metrics more often called Weyl fields and some work had been done on the stationary variety but no metric had appeared which was as physically appealing as that in (3.1). This metric is generally believed [15] to describe the exterior geometry of a slowly spinning spherical body with Schwarzschild mass m and angular momentum ma. This interpretation is a result of the weak field approximations first carried out by

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* The different forms of the Kerr metric along with the associated transformations may be found in appendix A.

Papapetrou [16].

The following is the usual weak field approximation technique found in standard texts on relativity. See for example Landau and Lifshitz [17]. First of all put

$$g_{ij} = \eta_{ij} + h^{(1)}_{ij} + h^{(2)}_{ij} \tag{3.2}$$

where

$$\eta_{ij} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{3.3}$$

$h^{(1)}_{ij}$ is the first approximation term, accurate to order $\frac{1}{r}$, while $h^{(2)}_{ij}$ is the second term accurate to order $\frac{1}{r^2}$. In the (t,x,y,z) coordinate system with $r^2 = x^2 + y^2 + z^2$ the h_{ij}'s are given by

$$\begin{aligned} h^{(1)}_{11} &= \frac{2m}{r} \\ h^{(1)}_{\alpha\beta} &= \frac{2m}{r} n_\alpha n_\beta \qquad \alpha, \beta = 2,3,4 \\ h^{(1)}_{1\alpha} &= 0 \qquad \alpha = 2,3,4 \end{aligned} \tag{3.4}$$

where $n_\alpha = \frac{x^\alpha}{r}$.

These are simply the first terms in the expansion of the Schwarzschild line element (1.4). The $h_{ij}^{(2)}$'s are given by

$$
\begin{aligned}
h_{11}^{(2)} &= 0 \\
h_{\alpha\beta}^{(2)} &= \frac{4m^2}{r^2} n_\alpha n_\beta \\
h_{1\alpha}^{(2)} &= \frac{2M_{\alpha\beta}}{r^2} n_\beta
\end{aligned}
\tag{3.5}
$$

with $\alpha, \beta = 2,3,4$. $M_{\alpha\beta}$ is the angular momentum three-tensor of the body together with its field. If we now consider the Kerr metric (1.9) and neglect terms of order a^2 we obtain

$$
\begin{aligned}
ds^2 \simeq &- r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \\
&- (1-\frac{2m}{r})^{-1} dr^2 + (1-\frac{2m}{r}) dt^2 \\
&- \frac{4ma}{r} \sin^2\theta d\phi dt,
\end{aligned}
\tag{3.6}
$$

and applying the transformation

$$
\begin{aligned}
r^2 &= x^2 + y^2 + z^2, \\
\phi &= \text{arc tan}\, \frac{y}{x}, \\
\theta &= \text{arc cos}\, \frac{z}{r},
\end{aligned}
\tag{3.7}
$$

we obtain

$$
\begin{aligned}
g_{11} &= 1 - \frac{2m}{r} \\
g_{\alpha\beta} &= \delta_{\alpha\beta} + \frac{2m}{r} n_\alpha n_\beta \\
g_{12} &= \frac{2may}{r^3} \qquad\qquad \alpha,\ \beta = 2,3,4 \\
g_{13} &= \frac{-2max}{r^3} \\
g_{14} &= 0
\end{aligned}
\tag{3.8}
$$

with $n_\alpha = \frac{x^\alpha}{r}$. Comparing (3.3), (3.4) with (3.8) we see that

$$g_{12} = \frac{2M_{23}y}{r^3} = \frac{2may}{r^3}$$

and (3.9)

$$g_{13} = \frac{2M_{32}x}{r^3} = \frac{-2max}{r^3}$$

from which we conclude that M_{23}, the angular momentum about the z axis, is equal to ma. Furthermore, by putting $a = 0$ in either (1.9) or (3.1) we obtain the Schwarzschild line element, which implies that m may be interpreted as Schwarzschild mass.

Perhaps the most elegant derivation of the Kerr metric was published by Ernst [18] who, unlike Kerr, proceeded from the

assumption of axial symmetry. The metric for a stationary axially symmetric vacuum field is given by

$$\begin{aligned} ds^2 &= g_{11}dt^2 + 2g_{14}d\phi dt + g_{22}dx^{2^2} \\ &\quad + 2g_{23}dx^2dx^3 + g_{33}dx^{3^2} + g_{44}d\phi^2 \end{aligned} \tag{3.10}$$

where it is assumed that the g_{ij} are independent of t and ϕ. This metric is by hypothesis invariant under a simultaneous inversion of t, ϕ. It was shown by Papapetrou [19] that (3.10) can be put in the form

$$\begin{aligned} ds^2 &= f^{-1}[e^{2\nu}(d\rho^2+dz^2)+\rho^2 d\phi^2] \\ &\quad -f(dt-\psi d\phi)^2 \end{aligned} \tag{3.11}$$

where $f = f(\rho,z)$, $\nu = \nu(\rho,z)$, $\psi = \psi(\rho,z)$. The vacuum field equations may be written as (the subscripts denote differentiation with respect to that variable)

$$\nu_\rho = \frac{\rho}{4f^2}(f_\rho^2-f_z^2) - \frac{f^2}{4\rho}(\psi_\rho^2-\psi_z^2) \tag{3.12a}$$

$$\nu_z = \frac{\rho}{2f^2} f_\rho f_z - \frac{f^2}{2\rho}\psi_\rho\psi_z \tag{3.12b}$$

$$f\nabla^2 f - (\nabla f)^2 = -\frac{f^4}{\rho^2}(\nabla\psi)^2 \tag{3.12c}$$

$$(\frac{f^2}{\rho}\psi_\rho)_\rho + (\frac{f^2}{\rho}\psi_z)_z = 0 \tag{3.12d}$$

where

$$\nabla^2 f = f_{\rho\rho} + \frac{1}{\rho} f_\rho + f_{zz}$$

$$(\nabla f)^2 = f_\rho^2 + f_z^2$$

(3.12c) and (3.12d) are integrability conditions for (3.12a) and (3.12b). It follows from (3.12d) that there exists a Φ such that

$$\frac{f^2}{\rho}\psi_\rho = \Phi_z, \quad \frac{f^2}{\rho}\psi_z = -\Phi_\rho \tag{3.13}$$

and hence we may write (3.12c) as

$$f\nabla^2 f - (\nabla f)^2 = -(\nabla\Phi)^2. \tag{3.14a}$$

Noting that $\psi_{\rho z} = \psi_{z\rho}$ we may rewrite (3.13) as

$$f\nabla^2\Phi = 2\nabla f\cdot\nabla\Phi \ . \tag{3.14b}$$

If we now define a complex function

$$\mathcal{E} = f + i\Phi$$
$$= \frac{\xi-1}{\xi+1} \tag{3.15}$$

equations (3.14a) and (3.14b) may be written as

$$(\xi\xi^*-1)\nabla^2\xi = 2\xi^*(\nabla\xi)^2 \tag{3.16}$$

where ξ^* is the complex conjugate of ξ . Introducing prolate spheroidal coordinates

$$\rho = (x^2-1)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}$$
$$z = xy \tag{3.17}$$

and rewriting (3.16) with the help of

$$\nabla^2 = \frac{1}{x^2-y^2}\left[\frac{\partial}{\partial x}(x^2-1)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}(1-y^2)\frac{\partial}{\partial y}\right] \tag{3.18}$$

$$\overline{\nabla}A\cdot\overline{\nabla}B = \frac{1}{x^2-y^2}\left[(x^2-1)\frac{\partial A}{\partial x}\frac{\partial B}{\partial x} + (1-y^2)\frac{\partial A}{\partial y}\frac{\partial B}{\partial y}\right]$$

it is noticed that $\xi = x$ is a solution of (3.16) . This corresponds to the Schwarzschild solution if we identify $r = x + 1$, $\cos\theta = y$ and length measured in units of m. Ernst pointed out that another solution of (3.16) is

$$\xi = x \cos\alpha + iy \sin\alpha, \tag{3.19}$$

where α is any real constant. Defining $\tan\alpha = a$, $\sec\alpha = m$, $r = x(m^2-a^2)^{\frac{1}{2}} + m$, lengths measured in units of $(m^2-a^2)^{\frac{1}{2}}$, and $\cos\theta = y$ we obtain , where

$\rho^2 = r^2 + a^2\cos^2\theta$

$$\begin{aligned} ds^2 = {} & \rho^2\left(d\theta^2 + \frac{dr^2}{r^2+a^2-2mr}\right) \\ & + (r^2+a^2)\sin^2\theta d\phi^2 - dt^2 \\ & + \frac{2\,mr}{\rho^2}(dt+a\sin^2\theta d\phi)^2 \end{aligned} \tag{3.20}$$

which is equal to metric (1.9) with a change in signature.

Another derivation of the Kerr metric was discovered by Newman and Janis [20]. They were able to obtain the Kerr line element by a series of complex transformations on the contravariant components of the metric tensor for the Schwarzschild line element. A similar transformation [21] was applied to the Reisnner-Nordström metric to obtain a "charged-Kerr" metric

$$ds^2 = \left(1+\frac{\varepsilon^2-2mr}{\rho^2}\right)du^2 + 2dudr$$

$$+ \frac{2(a\sin^2\theta)(2mr-\varepsilon^2)}{\rho^2}\,dud\phi \qquad (3.21)$$

$$- 2a\sin^2\theta\, drd\phi - \rho^2 d\theta^2$$

$$- \sin^2\theta\left(r^2+a^2+\frac{a^2\sin^2\theta(2mr-\varepsilon^2)}{\rho^2}\right) d\phi^2$$

with $\rho^2 = r^2 + a^2\cos^2\theta$.

It is interesting to note that the authors can give no clear, simple reason as to why the series of transformations should yield any solution to the field equations, much less the Kerr or "charged-Kerr" metrics.

CHAPTER IV

THE "KERR-VAIDYA" METRIC

It was seen in the analysis of the Vaidya solution that in order for radiation to occur we must have a nonstatic line element where $m = m(r,t)$ or, in radiation coordinates (u,r,θ,ϕ), $m = m(u)$, where u is retarded time. As it is our hope to obtain a "radiating Kerr metric", that is, the exterior metric of a radiating rotating body, it was thought that we should also assume that $a = a(u)$. That is, we should allow for possible variation in the angular momentum per unit mass.

The field equations were hoped to be some generalization of the Vaidya equation

$$R_{ij} = -\ 8\pi q v_i v_j , \tag{4.1}$$

where $v_i v^i = 0$. However, this turned out not to be the case. Preliminary calculations based on the metric

$$\begin{aligned} ds^2 = {} & (1 - \frac{2mr}{\rho^2}) du^2 + 2dudr \\ & + \frac{4mra\ \sin^2\theta}{\rho^2}\ dud\phi \\ & - 2a\ \sin^2\theta drd\phi - \rho^2 d\theta^2 \\ & - \sin^2\theta(r^2 + a^2 + \frac{2mra^2\sin^2\theta}{\rho^2}) d\phi^2 \end{aligned} \tag{4.2}$$

with $\rho^2 = r^2 + a^2\cos^2\theta$, $m = m(u)$ and $a = a(u)$ led to the two equations

$$R_{22} = 0 \tag{4.3a}$$

$$R_{23} = -\frac{2aa'r^2\sin\theta\cos\theta}{\rho^4} . \tag{4.3b}$$

Equation (4.3a) implies that $v_2 = 0$, which by (4.1) implies that $R_{23} = 0$. However, this is not the case unless we assume $a' = 0$. Even though some generality is lost by this assumption it has a further attraction in that it makes the computation of the field equations far less prohibitive.

The condition $a' = 0$ seems to imply that we are considering only those bodies whose angular velocity and radius are unaffected by the radiation. This follows from the fact that rotational inertia of a spherical body is proportional to mR^2, m being the mass of the body and R the radius. Therefore, the angular momentum is proportional to mR^2w where w is the angular velocity and hence

$$a \propto R^2w . \tag{4.4}$$

A model might be a hollow, radiating sphere that has a constant angular momentum per unit mass, but as its density need not remain constant it could still realize a net mass change.

The remaining components of the Ricci tensor were calculated using metric (4.2) with the assumption $a' = 0$. We refer the reader to appendix B for the Christoffel symbols and other calculations and note the results here.

The metric tensor in radiation coordinates is

$$g_{ij} = \begin{pmatrix} 1 - \frac{2mr}{\rho^2} & 1 & 0 & \frac{2mra\sin^2\theta}{\rho^2} \\ 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & -\rho^2 & 0 \\ \frac{2mra\sin^2\theta}{\rho^2} & -a\sin^2\theta & 0 & -\sin^2\theta(r^2+a^2+\frac{2mra^2\sin^2\theta}{\rho^2} \end{pmatrix} \tag{4.5a}$$

$$g = \det g_{ij} = -\rho^4\sin^2\theta \tag{4.5b}$$

and

$$g^{ij} = \begin{pmatrix} \frac{-a^2\sin^2\theta}{\rho^2} & \frac{r^2+a^2}{\rho^2} & 0 & \frac{-a}{\rho^2} \\ \frac{r^2+a^2}{\rho^2} & \frac{2mr-(r^2+a^2)}{\rho^2} & 0 & \frac{a}{\rho^2} \\ 0 & 0 & -\frac{1}{\rho^2} & 0 \\ \frac{-a}{\rho^2} & \frac{a}{\rho^2} & 0 & \frac{-1}{\rho^2\sin^2\theta} \end{pmatrix}, \tag{4.5c}$$

from which we calculated

$$R_{11} = m''\left(\frac{ra^2\sin^2\theta}{\rho^4}\right) + m' \frac{2r^2(r^2+a^2)}{\rho^6} , \tag{4.6a}$$

$$R_{14} = (-a \sin^2\theta)\left[R_{11} + \frac{m'(r^2-a^2\cos^2\theta)}{\rho^4}\right], \tag{4.6b}$$

$$R_{44} = (a^2\sin^4\theta)\left[R_{11} + \frac{2m'(r^2-a^2\cos^2\theta)}{\rho^4}\right], \tag{4.6c}$$

$$R_{13} = \frac{2m'ra^2\sin\theta\cos\theta}{\rho^4} , \tag{4.6d}$$

$$R_{34} = (-a \sin^2\theta)R_{13}, \tag{4.6e}$$

$$R_{12} = R_{22} = R_{32} = R_{33} = R_{24} = 0. \tag{4.6f}$$

Equations (4.6) may be written compactly in tensorial notation as

$$R_{ij} = - 8\pi q(w_i w_j + w_i a_j + a_i w_j) \tag{4.7}$$

where

$$w_i = (1,0,0,-a \sin^2\theta) \tag{4.8a}$$

$$a_i = (0,0,\alpha,-\beta a \sin^2\theta) \tag{4.8b}$$

and

$$\alpha = -\frac{1}{8\pi q} R_{13} \tag{4.8c}$$

$$\beta = -\frac{1}{8\pi q} \frac{m'(r^2-a^2\cos^2\theta)}{\rho^4} \tag{4.8d}$$

It follows that

$$w^i = g^{ij}w_j = (0,1,0,0). \tag{4.9}$$

Hence

$$w^i w_i = 0, \quad w^i a_i = 0, \tag{4.10}$$

and we conclude that

$$R = -8\pi q g^{ij}(w_i w_j + w_i a_j + a_i w_j) = 0. \tag{4.11}$$

This result implies the energy tensor has the form

$$T_{ij} = q(w_i w_j + w_i a_j + a_i w_j). \tag{4.12}$$

Another immediate observation is that if $a \to 0$ then metric (4.2) approached the Vaidya metric. But when $a \cong 0$ we have that

$$a_i \overset{\sim}{=} 0 \tag{4.13a}$$

$$w_4 \overset{\sim}{=} 0 \tag{4.13b}$$

and equations (4.7) reduce to

$$R_{ij} = -\, 8\pi q w_i w_j \tag{4.14}$$

with $w_i = (1,0,0,0)$, which is Vaidya's form of the field equations.

CHAPTER V

SOME PHYSICAL ASPECTS OF THE

"KERR-VAIDYA" METRIC

In this chapter we will be concerned primarily with certain physical interpretations of the radiating Kerr metric where we assume that a is small. One reason for this assumption is that at a sufficiently large distance from the rotating body we expect the radiation to be directed. This suggests that the field equations should be asymptotic to the form

$$R_{ij} = - 8\pi q v_i v_j, \tag{5.1}$$

where $v^i v_i = 0$. This form may be obtained by neglecting terms of order a^2 in equation (4.7). That is, we may write (to first order in a)

$$w_i w_j + w_i a_j + a_i w_j = (w_i + a_i)(w_j + a_j) \tag{5.2}$$

and define

$$v_i = w_i + a_i = (1,0,0, \frac{-3}{2} a \sin^2\theta). \tag{5.3}$$

Since v^i has the form

$$v^i = (0,1,0,\frac{1}{2}\frac{a}{r^2}) \tag{5.4}$$

it follows that

$$v^i v_i = 0 . \tag{5.5}$$

Therefore, in the following analysis, we will use the "approximate" form

$$R_{ij} = - 8\pi q v_i v_j \tag{5.6}$$

where

$$v_i = (1,0,0,\frac{-3}{2} a \sin^2\theta)$$

instead of field equations (4.7).

The first problem we consider is to find the effect that the rotation has on the path of a photon. The form of (5.6) implies that the photon path is given by equations of the form

$$x^i = x^i(\xi), \qquad \frac{dx^i}{d\xi} \sim v^i . \tag{5.7}$$

The solution to (5.7) is evidently a null curve; the fact that it is indeed a null geodesic can be shown as follows. Geodesics are determined by the equation

$$\frac{d^2x^i}{d\xi^2} + \left\{ {i \atop jk} \right\} \frac{dx^j}{d\xi} \frac{dx^k}{d\xi} = \phi(\xi) \frac{dx^i}{d\xi} \tag{5.8}$$

which may be written as

$$\frac{\delta}{\delta\xi} \left(\frac{dx^i}{d\xi}\right) = \phi \frac{dx^i}{d\xi} \tag{5.9}$$

But

$$\frac{\delta}{\delta\xi} \left(\frac{dx^i}{d\xi}\right) = \frac{\delta}{\delta\xi} v^i = v^i_{;j} v^j \tag{5.10}$$

hence the equation for a null geodesic is equivalent to the condition

$$v^i_{;j} v^j = \phi(\xi) v^i . \tag{5.11}$$

However, from the conservation equation

$$T^i_{j;i} = 0 \tag{5.12}$$

we obtain, since $T^i_j = qv^i v_j$,

$$(qv^i)_{;i}v_j = -qv^i v_{j;i} \tag{5.13}$$

or

$$v^i v_{j;i} = \phi v_j \tag{5.14}$$

which of course implies (5.9).

The solution to the differential equation

$$\frac{dx^i}{d\xi} = (0,1,0,\ \frac{1}{2}\frac{a}{r^2}) \tag{5.15}$$

has the form

$$\begin{aligned} u &= c_1 & r &= \xi \\ \theta &= c_2 & \phi &= c_3 - \frac{a}{2r}\ . \end{aligned} \tag{5.16}$$

Let us now consider a photon leaving the surface of the star, say, at the point

$$r = R, \quad \phi = 0, \quad \theta = \frac{\pi}{2}\ .$$

The trajectory of the photon is described by the equation

$$r = \frac{R\alpha}{\alpha-\phi} \tag{5.17}$$

where $\alpha = \frac{a}{2R}$. If we put

$$x = r\cos\phi, \quad y = r\sin\phi$$

we have that

$$\frac{dy}{dx} = \frac{\tan\phi - (\phi-\alpha)}{1 + (\phi-\alpha)\tan\phi} \tag{5.18}$$

and

$$\frac{d}{d\phi}\left(\frac{dy}{dx}\right) = \left\{\frac{(\phi-\alpha)\sec\phi}{1 + (\phi-\alpha)\tan\alpha}\right\}^2 > 0 \tag{5.19}$$

Therefore, $\frac{dy}{dx}$ is an increasing function of ϕ . Since

$$\left(\frac{dy}{dx}\right)_{\phi=o} = \alpha \tag{5.20a}$$

$$\left(\frac{dy}{dx}\right)_{\phi=\alpha} = \tan\alpha > \alpha \tag{5.20b}$$

we have that the trajectory of the photon is concave upward and asymptotic to a line with slope $\tan\alpha$. (see diagram 5.1).

$$u_3 = 0 \tag{5.26c}$$

$$u_4 = \frac{2ma\ \sin^2\theta}{r}\frac{du}{d\tau} - aU\ \sin^2\theta \ - \frac{a\ \sin^2\theta}{2}\ U \tag{5.26d}$$

where we have neglected $0(a^2)$. From the normalizing condition $u^\mu u_\mu = 1$ we obtain that

$$\frac{du}{d\tau} = \frac{1}{\alpha + U} \tag{5.27}$$

where

$$\alpha = (U^2 + 1 - \frac{2m}{r})^{\frac{1}{2}}$$

and hence, since $\bar{q} = u^i u^j T_{ij}$, we have that

$$\bar{q} = u^1 u^1 T_{11} = (\frac{1}{\alpha+U})^2 q. \tag{5.28}$$

If we again define luminosity as in (2.29) we have that

$$\begin{aligned} L_\infty &= \lim_{\substack{r \to \infty \\ U \to o}} 4\pi r^2 \bar{q} \\ &= -m', \end{aligned} \tag{5.29}$$

the amount of mass radiated per unit time.

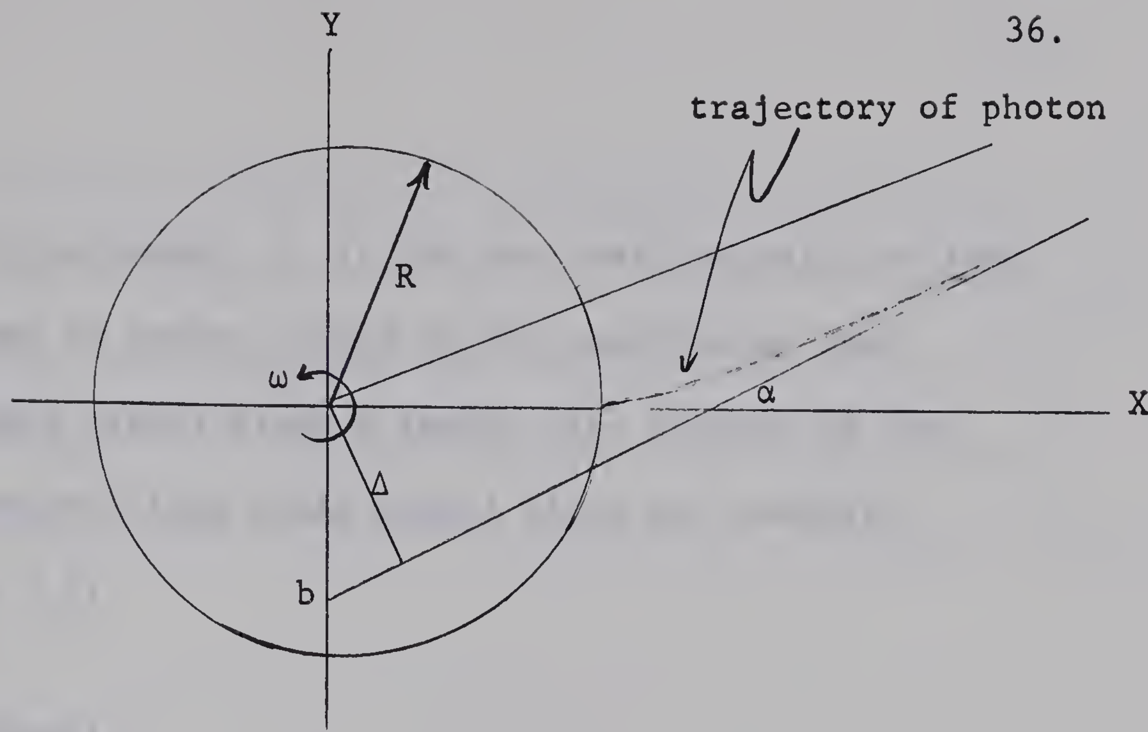

Diagram 5.1.

Suppose the equation of the asymptote is

$$y = (\tan \alpha)x + b, \tag{5.21}$$

then for large x and y we have that

$$r = [(x \tan \alpha + b)^2 + x^2]^{\frac{1}{2}} = \frac{R\alpha}{\alpha - \text{Arctan}\, \frac{y}{x}} \tag{5.22}$$

from which, letting $x \to \infty$, we obtain

$$b = \frac{-a}{2} \sec \alpha \,. \tag{5.23}$$

Equation (5.23) enables us to calculate the "apparent deflection" Δ . That is, a distant observer would see the

star shift by an amount Δ if the star was initially at rest and then began to rotate, since in the nonrotating case the light would travel along a radial line whereas in the rotating case the light would travel along our geodesic. (see Diagram 5.1).

Evidently

$$\begin{aligned} \Delta &= (-b) \cos \alpha \\ &= \frac{a}{2} . \end{aligned} \tag{5.24}$$

We now define as in the Vaidya analysis $\overline{q}$ to be the energy density of the radiation measured locally by an observer moving with four velocity u^μ. For convenience we choose

$$u^\mu = (\frac{du}{d\tau} , U, 0, \frac{a}{2r^2} U) \tag{5.25}$$

where $U = \frac{dr}{d\tau}$.

Hence

$$u_1 = (1 - \frac{2m}{r}) \frac{du}{d\tau} + U \tag{5.26a}$$

$$u_2 = \frac{du}{d\tau} \tag{5.26b}$$

We now wish to calculate the total angular momentum in the system using the pseudo-tensors given in Landau and Lifshitz [22]. We first, however, must transform the radiating Kerr metric (4.2) to an asymptotically flat coordinate system. The transformation that will do this is given by Kerr (see appendix A, equations (A.2) (A.3))

$$(r-ia)e^{i\phi}\sin\theta = x + iy$$
$$r\cos\theta = z. \tag{5.30}$$
$$u = t - r.$$

The resulting metric, neglecting terms of order a^2, is

$$ds^2 = -(dx^2+dy^2+dz^2) + dt^2$$
$$-\frac{2m}{r^3}\left[xdx+ydy+zdz-rdt+\frac{a}{r}(xdy-ydx)\right]^2 \tag{5.31}$$

where $$r^2 = x^2 + y^2 + z^2.$$

This metric may be written as

$$ds^2 = (1-\frac{2m}{r})dt^2 - (1+\frac{2mx}{r^3} - \frac{4mayx}{r^4})dx^2 \tag{5.32}$$

$$-(1+\frac{2my^2}{r^3} + \frac{4maxy}{r^4})dy^2 - (1+\frac{2mz^2}{r^3})dz^2$$

$$+2(\frac{2mx}{r^2} - \frac{2may}{r^3})dxdt + 2(\frac{2my}{r^2} + \frac{2max}{r^3})dydt$$

$$+ 2(\frac{2mz}{r^2})dzdt + 2(-\frac{2xym}{r^3} - \frac{2ma(x^2-y^2)}{r^4})dxdy$$

$$+ 2(\frac{-2mzx}{r^3} + \frac{2mayz}{r^4})dxdz$$

$$+ 2(\frac{-2mzy}{r^3} - \frac{2maxz}{r^4})dydz.$$

Hence to the first order in a we have

$$g_{ij} = \begin{pmatrix} 1 - \frac{2m}{r} & \frac{2m}{r^2}(x-\frac{ay}{r}) & \frac{2m}{r^2}(y+\frac{ax}{r}) & \frac{2mz}{r^2} \\ \frac{2m}{r^2}(x-\frac{ay}{r}) & -1-\frac{2m}{r^3}(x^2-\frac{2ayx}{r}) & \frac{-2m}{r^3}[xy+\frac{a(x^2-y^2)}{r}] & \frac{-2mz}{r^3}(x-\frac{ay}{r}) \\ \frac{2m}{r^2}(y+\frac{ax}{r}) & \frac{-2m}{r^3}[xy+\frac{a(x^2-y^2)}{r}] & -1-\frac{2m}{r^3}(y^2+\frac{2ayx}{r}) & \frac{-2mz}{r^3}(y+\frac{ax}{r}) \\ \frac{2mz}{r^2} & \frac{-2mz}{r^3}(x-\frac{ay}{r}) & \frac{-2mz}{r^3}(y+\frac{ax}{r}) & -1-\frac{2mz^2}{r^3} \end{pmatrix} \tag{5.33a}$$

$$g = \det g_{ij} = -1 \tag{5.33b}$$

$$g^{ij} = \begin{pmatrix} 1 + \frac{2m}{r} & \frac{2m}{r^2}(x-\frac{ay}{r}) & \frac{2m}{r^2}(y+\frac{ax}{r}) & \frac{2mz}{r^2} \\ \frac{2m}{r^2}(x-\frac{ay}{r}) & -1+\frac{2mx}{r^3}(x-\frac{2ay}{r}) & \frac{2mxy}{r^3}+\frac{ma(x^2-y^2)}{r^4} & \frac{2mz}{r^3}(x-\frac{ay}{r}) \\ \frac{2m}{r^2}(y+\frac{ax}{r}) & \frac{2mxy}{r^3}+\frac{ma(x^2-y^2)}{r^4} & -1+\frac{2my}{r^3}(y+\frac{2ax}{r}) & \frac{2mz}{r^3}(y+\frac{ax}{r}) \\ \frac{2mz}{r^2} & \frac{2mz}{r^3}(x-\frac{ay}{r}) & \frac{2mz}{r^3}(y+\frac{ax}{r}) & -1+\frac{2mz^2}{r^3} \end{pmatrix} \tag{5.33c}$$

If we now define

$$\begin{aligned} \nu^\alpha &= (x - \frac{ay}{r} , \quad y + \frac{ax}{r} , \ z) \\ &= x^\alpha + \frac{au^\alpha}{r} \end{aligned} \tag{5.34}$$

where $x^\alpha = (x,y,z)$, $u^\alpha = (-y,x,0)$, (Greek letters in the following analysis take on the values 2,3,4), we obtain

$$g^{\alpha\beta} = \delta^{\alpha\beta} + \frac{2m}{r^3} \nu^\alpha \nu^\beta \tag{5.35a}$$

$$g^{1\alpha} = \frac{2m}{r} \nu^\alpha \tag{5.35b}$$

$$g^{11} = 1 + \frac{2m}{r} . \tag{5.35c}$$

We are now in a position to determine the angular momentum. By definition [23] the angular momentum three tensor is given by

$$M^{\alpha\beta} = \int_S (x^\alpha h^{\beta 1\gamma} - x^\beta h^{\alpha 1\gamma} + \lambda^{\alpha 1\gamma\beta}) df_\gamma \tag{5.36}$$

where

$$\begin{aligned} h^{\beta 1\gamma} &= \frac{1}{16\pi} [(-g)(g^{\beta 1} g^{\gamma m} - g^{\beta\gamma} g^{1m}]_{,m} \\ &= \lambda^{\beta 1\gamma m}{}_{,m} \end{aligned} \tag{5,37}$$

where $m = 1,2,3,4$.

The surface of integration in (5.36), for our case, will be that of a large sphere so that $df_\alpha = n_\alpha r^2 \sin\theta d\theta d\phi$ and $n_\alpha = \frac{x^\alpha}{r}$. Substituting (5.35) and (5.37) into (5.36) we obtain

$$M^{\alpha\beta} = \frac{1}{16\pi}\int_S \Big[\frac{2ma}{r^3}(\delta^{\alpha\gamma}u^\beta - \delta^{\beta\gamma}u^\alpha) + \frac{4ma}{r^5}x^\gamma(x^\alpha u^\beta - x^\beta u^\alpha) - \frac{2m}{r^2}(x^\alpha \nu^\beta{}_{,\gamma} - x^\beta \nu^\alpha{}_{,\gamma})\Big]df_\gamma \tag{5.38}$$

(we sum over $\gamma = 2,3,4$).

If we now consider M^{23}, the angular momentum about the z axis, we have that

$$M^{23} = \frac{1}{16\pi}\int_S \Big[\frac{2ma}{r^3}\Big(\delta^{2\gamma}x + \delta^{3\gamma}y + \frac{2x^\gamma}{r^2}(x^2+y^2)\Big) - \frac{2m}{r^2}(x\nu^3{}_{,\gamma} - y\nu^2{}_{,\gamma})\Big]df_\gamma . \tag{5.39}$$

Now applying equations (5.34) and the relation $df_\alpha = \frac{x^\alpha}{r}dS$, we reduce (5.39) to

$$M^{23} = \frac{1}{16\pi}\int_S \frac{6ma(x^2+y^2)}{r^4}\, dS. \tag{5.40}$$

A straight forward calculation shows that

$$M^{23} = ma.$$

On the other hand,

$$M^{24} = \frac{1}{16\pi} \int_S \left[\frac{2ma}{r^3} (y\delta^{4\gamma}) - \frac{4ma}{r^5} x^\gamma (yz) - \frac{2m}{r^2} (x\nu^4_{,\gamma} - z\nu^2_{,\gamma})\right] df_\gamma \tag{5.41}$$

which reduces to

$$M^{24} = \frac{1}{16\pi} \int_S \frac{6mayz}{r^4} dS = 0. \tag{5.42}$$

It may be shown that M^{23} is in fact the only nonzero component of angular momentum three-tensor.

The above results have the following implications. If R is sufficiently large so that terms of order $\frac{1}{R}$ are negligible with respect to unity then the sphere of integration $x^2 + y^2 + z^2 = R^2$ contains angular momentum described by the three vector

$$H_\alpha = (0,0,ma) \tag{5.43}$$

where $m = m(t-R)$. At a time Δt later, H_4 will be reduced because m is assumed to be a decreasing function. Evidently, the angular momentum radiated per unit time will be

$$\lim_{\Delta t \to o} \frac{m(t-R)a - m(t+\Delta t-R)a}{\Delta t} = -m'a. \tag{5.44}$$

We now wish to turn our attention to a particular model which might exhibit a radiating Kerr metric. It has been pointed out by De la Cruz and Israel [24] that a non radiating rotating shell is a possible source of the Kerr metric. This is substantiated by the fact that they were able to join the Kerr metric continuously, in a first approximation, to a flat interior line element. A similar analysis will show that the radiating Kerr metric may also be joined continuously to a flat, interior line element.

The radiating Kerr metric (4.2) induces, neglecting orders of a^2, the intrinsic metric

$$(ds^2)_\Sigma = (1 - \frac{2m(u)}{R})du^2 + \frac{4ma \sin^2\theta}{R} du d\phi$$

$$- R^2 d\theta^2 - R^2 \sin^2\theta d\phi^2 \tag{5.45}$$

on the hypersurface $\sum : r = R$. A flat interior metric may be written as

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\Phi^2 + dt^2 \tag{5.46}$$

which reduces to

$$(ds^2)_\Sigma = -R^2(d\theta^2 + \sin^2\theta d\Phi^2) + dt^2 \tag{5.47}$$

on $\sum : r = R$.

If we now define

$$\Phi = \phi + \int \frac{2m(u)a}{R^3} du \tag{5.48}$$

$$t = \int (1 - \frac{2m(u)}{R}) du$$

metric (5.47) transforms into metric (5.45).

In other words the above pair of compatible metrics might be considered as describing the field of a slowly rotating "insulated" spherical shell. The radiation is from the outer boundary only, as the flat interior would require that there be at most a negligible amount of radiation on the inside of the sphere.

APPENDIX A

FORMS OF KERR METRIC

The metric first given by Kerr [25] in 1963 is

$$
\begin{aligned}
ds^2 &= \rho^2(d\theta^2+\sin^2\theta d\phi^2) \\
&+ 2(du+a\sin^2\theta d\phi)(dr+a\sin^2\theta d\phi) \\
&- (1-\frac{2mr}{\rho^2})(du+a\sin^2\theta d\phi)^2
\end{aligned}
\tag{A.1}
$$

where $\rho^2 = r^2 + a^2\cos^2\theta$.

This metric differs from metric (4.2), as well as the one given by Newman et al [26], by a change in signature and the transformation

$$
\begin{aligned}
u &\to -\bar{u} \\
t &\to -\bar{t}
\end{aligned}
\tag{A.2}
$$

and it differs from the metric given by Carter [27] by a change in the sign of a. Kerr points out in that paper that metric (A.1) may be transformed to an asymptotically flat coordinate system by the transformation

$$
\begin{aligned}
(r-ia)e^{i\phi}\sin\theta &= x + iy \\
r\cos\theta &= z \\
u &= t + r
\end{aligned}
\tag{A.3}
$$

the metric becoming

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2$$

$$+ \frac{2mr^3}{r^4+a^2z^2}\left[\frac{r(xdx+ydy)+a(xdy-ydx)}{(r^2+a^2)} + \frac{zdz+rdt}{r}\right]^2 \qquad (A.4)$$

where r is defined by the equation

$$r^4 - (x^2+y^2+z^2-a^2)r^2 - a^2z^2 = 0 \qquad (A.5)$$

Boyer and Lindquist [28] apply the transformation

$$x = (r^2+a^2)^{\frac{1}{2}}\sin\theta \cos(\phi - \tan^{-1}\frac{a}{r})$$

$$y = (r^2+a^2)^{\frac{1}{2}}\sin\theta \sin(\phi - \tan^{-1}\frac{a}{r}) \qquad (A.6)$$

$$z = r\cos\theta$$

to metric (A.4) and obtain

$$ds^2 = dr^2 + 2a\sin^2\theta drd\phi + (r^2+a^2)\sin^2\theta d\phi^2$$

$$+ \rho^2 d\theta^2 - dt^2 + \frac{2mr}{\rho^2}(dr+a\sin^2\theta d\phi+dt)^2 \qquad (A.7)$$

They further present the transformation

$$\bar{r} = r, \quad \bar{\theta} = \theta$$

$$\overline{d\phi} = d\phi + a \frac{dr}{r^2 - 2mr + a^2} \tag{A.8}$$

$$\overline{dt} = dt - 2mr \frac{dr}{r^2 - 2mr + a^2}$$

which will take metric (A.7) into

$$ds^2 = \rho^2 \left(\frac{dr^2}{r^2-2mr+a^2} + d\theta^2\right) + (r^2+a^2)\sin^2\theta \; \overline{d\phi}^2$$
$$- \overline{dt}^2 + \frac{2mr}{\rho^2}\left(a \sin^2 \overline{d\phi} + \overline{dt}\right)^2 . \tag{A.9}$$

The $(r,\theta,\bar{\phi},\bar{t})$ coordinates are referred to as Schwarzschild coordinates, as metric (A.9) reduces to the standard Schwarzschild metric when $a = 0$.

APPENDIX B

CALCULATION OF THE FIELD EQUATIONS

FOR THE "KERR-VAIDYA" METRIC

The metric tensor for the "Kerr-Vaidya" line element is given by equations (4.5). However, in the following calculations it was advantageous to use the following form

$$g_{ij} = \begin{pmatrix} 1+\alpha & 1 & 0 & \alpha\beta \\ 1 & 0 & 0 & \beta \\ 0 & 0 & -\rho^2 & 0 \\ \alpha\beta & \beta & 0 & -\beta^2(1-\alpha+\gamma^2) \end{pmatrix} \tag{B.1a}$$

$$g = \text{determinant } g_{ij} = -\beta^2\rho^2\gamma^2 \tag{B.1b}$$

$$g^{ij} = \begin{pmatrix} \gamma^{-2} & (1+\gamma^2)\gamma^{-2} & 0 & (\beta\gamma^2)^{-1} \\ (1+\gamma^2)\gamma^{-2} & -((1+\gamma^2)\gamma^{-2}+\alpha) & 0 & (-\beta\gamma^2)^{-1} \\ 0 & 0 & \rho^{-2} & 0 \\ (\beta\gamma^2)^{-1} & (-\beta\gamma^2)^{-1} & 0 & -(\beta\gamma)^{-2} \end{pmatrix} \tag{B.1c}$$

where
$$\rho^2 = r^2 + a^2\cos^2\theta,$$
$$\beta = -a\sin^2\theta, \tag{B.2}$$
$$\alpha = \frac{-2mr}{\rho^2}, \quad \gamma^2 = \frac{\rho^2}{a^2\sin^2\theta}.$$

The Christoffel symbols $\{{}^{\,i}_{j\,k}\}$ are given by

$$\{{}^{\,i}_{j\,k}\} = g^{im}[jk,m] \tag{B.3}$$

with $$[jk,m] = \frac{1}{2}\,[g_{jm,k}+g_{km,j}-g_{jk,m}]$$

where $$g_{jk,m} = \frac{\partial}{\partial x^m}\, g_{jk} \;.$$

The Ricci tensor $R_{j\ell}$ is given by

$$\begin{aligned} R_{j\ell} &= R^{i}_{ji\ell} \\ &= [\log\,(-g)^{\frac{1}{2}}]_{,j\ell} - \{{}^{\,i}_{j\,\ell}\}_{,i} \\ &\quad + \{{}^{\,i}_{m\,\ell}\}\{{}^{\,m}_{i\,j}\} - [\log\,(-g)^{\frac{1}{2}}]_{,i}\{{}^{\,i}_{\ell\,j}\}. \end{aligned} \tag{B.4}$$

We did not calculate each R_{ij} explicitly but rather wrote the Christoffel symbols in the form

$$\{{}^{\,i}_{j\,k}\} = \{{}^{\,i}_{j\,k}\}^{*} + \Gamma^{i}_{jk} \tag{B.5}$$

where $\{{}^{\,i}_{j\,k}\}^{*}$ are the Christoffel symbols for the usual Kerr metric in which m is constant. The terms Γ^{i}_{jk} involved terms depending on m' and m''. It follows that

$$R_{ij} = R^{*}_{ij} + \Gamma_{ij} \tag{B.6}$$

where R^*_{ij} is the Ricci tensor formed from the $\{{}^{\,i}_{j\,k}\}^*$ while Γ_{ij} involves the terms Γ^i_{jk} . Since $R^*_{ij} = 0$ we have that

$$R_{ij} = \Gamma_{ij} \ . \tag{B.7}$$

Hence we may ignore the "starred quantities" where they do not affect terms involving m' or m''.

The Christoffel symbols are

$$\{{}^{\,1}_{1\,1}\} = \{{}^{\,1}_{1\,1}\}^* - \frac{m' r a^2 \sin^2\theta}{\rho^4}$$

$$\{{}^{\,1}_{1\,2}\} = 0$$

$$\{{}^{\,1}_{1\,3}\} = \{{}^{\,1}_{1\,3}\}^*$$

$$\{{}^{\,1}_{1\,4}\} = \{{}^{\,1}_{1\,4}\}^* + \frac{m' r a^3 \sin^4\theta}{\rho^4}$$

$$\{{}^{\,1}_{2\,2}\} = 0$$

$$\{{}^{\,1}_{2\,3}\} = \{{}^{\,1}_{2\,3}\}^*$$

$$\{{}^{\,1}_{2\,4}\} = \{{}^{\,1}_{2\,4}\}^*$$

$$\{{}^{\,1}_{3\,3}\} = \{{}^{\,1}_{3\,3}\}^*$$

$$\left\{{1 \atop 3\ 4}\right\} = \left\{{1 \atop 3\ 4}\right\}^*$$

$$\left\{{1 \atop 4\ 4}\right\} = \left\{{1 \atop 4\ 4}\right\}^* - \frac{m'ra^4\sin^6\theta}{\rho^4}$$

$$\left\{{2 \atop 1\ 1}\right\} = \left\{{2 \atop 1\ 1}\right\}^* + \frac{m'ra^2\sin^2\theta}{\rho^4} - \frac{m'r}{\rho^2}$$

$$\left\{{2 \atop 1\ 2}\right\} = \left\{{2 \atop 1\ 2}\right\}^*$$

$$\left\{{2 \atop 1\ 3}\right\} = 0$$

$$\left\{{2 \atop 1\ 4}\right\} = \left\{{2 \atop 1\ 4}\right\}^* - \frac{m'ra^3\sin^4\theta}{\rho^4}$$

$$\left\{{2 \atop 2\ 2}\right\} = 0$$

$$\left\{{2 \atop 2\ 3}\right\} = \left\{{2 \atop 2\ 3}\right\}^*$$

$$\left\{{2 \atop 2\ 4}\right\} = \left\{{2 \atop 2\ 4}\right\}^*$$

$$\left\{{2 \atop 3\ 3}\right\} = \left\{{2 \atop 3\ 3}\right\}^*$$

$$\left\{{2 \atop 3\ 4}\right\} = \left\{{2 \atop 3\ 4}\right\}^*$$

$$\left\{{2 \atop 4\ 4}\right\} = \left\{{2 \atop 4\ 4}\right\}^* + \frac{m'ra^2\sin^4\theta}{\rho^2}\left(1 + \frac{a^2\sin^2\theta}{\rho^2}\right)$$

$$\left\{{3 \atop 1\ 1}\right\} = \left\{{3 \atop 1\ 1}\right\}^*$$

$$\left\{ {3 \atop 1\;2} \right\} = 0$$

$$\left\{ {3 \atop 1\;3} \right\} = 0$$

$$\left\{ {3 \atop 1\;4} \right\} = \left\{ {3 \atop 1\;4} \right\}^*$$

$$\left\{ {3 \atop 2\;2} \right\} = 0$$

$$\left\{ {3 \atop 2\;3} \right\} = \left\{ {3 \atop 2\;3} \right\}^*$$

$$\left\{ {3 \atop 2\;4} \right\} = \left\{ {3 \atop 2\;4} \right\}^*$$

$$\left\{ {3 \atop 3\;3} \right\} = \left\{ {3 \atop 3\;3} \right\}^*$$

$$\left\{ {3 \atop 3\;4} \right\} = 0$$

$$\left\{ {3 \atop 4\;4} \right\} = \left\{ {3 \atop 4\;4} \right\}^*$$

$$\left\{ {4 \atop 1\;1} \right\} = \left\{ {4 \atop 1\;1} \right\}^* - \frac{m'ra}{\rho^4}$$

$$\left\{ {4 \atop 1\;2} \right\} = \left\{ {4 \atop 1\;2} \right\}^*$$

$$\left\{ {4 \atop 1\;3} \right\} = \left\{ {4 \atop 1\;3} \right\}^*$$

$$\left\{ {4 \atop 1\;4} \right\} = \left\{ {4 \atop 1\;4} \right\}^* + \frac{m'ra^2\sin^2\theta}{\rho^4}$$

$$\{{}_{2}{}^{4}{}_{2}\} = 0$$

$$\{{}_{2}{}^{4}{}_{3}\} = \{{}_{2}{}^{4}{}_{3}\}^*$$

$$\{{}_{2}{}^{4}{}_{4}\} = \{{}_{2}{}^{4}{}_{4}\}^*$$

$$\{{}_{3}{}^{4}{}_{3}\} = \{{}_{3}{}^{4}{}_{3}\}^*$$

$$\{{}_{3}{}^{4}{}_{4}\} = \{{}_{3}{}^{4}{}_{4}\}^*$$

$$\{{}_{4}{}^{4}{}_{4}\} = \{{}_{4}{}^{4}{}_{4}\}^* - \frac{m'ra^3\sin^4\theta}{\rho^4} .$$

The "starred" Christoffel symbols are

$$\{{}_{1}{}^{1}{}_{1}\}^* = \frac{-m(r^2+a^2)(r^2-a^2\cos^2\theta)}{\rho^6}$$

$$\{{}_{1}{}^{1}{}_{2}\}^* = 0$$

$$\{{}_{1}{}^{1}{}_{3}\}^* = \frac{-2ma^2r\sin\theta\cos\theta}{\rho^4}$$

$$\{{}_{1}{}^{1}{}_{4}\}^* = \frac{am(r^2+a^2)\sin^2\theta(r^2-a^2\cos^2\theta)}{\rho^6}$$

$$\{{}_{2}{}^{1}{}_{2}\}^* = 0$$

$$\{{}_{2}{}^{1}{}_{3}\}^* = \frac{a^2\sin\theta\cos\theta}{\rho^2}$$

$$\{{}_{2}{}^{1}{}_{4}\}^* = \frac{ra\sin^2\theta}{\rho^2}$$

$$\left\{{}_{3\ 3}^{\ 1}\right\}^* = \frac{r(r^2+a^2)}{\rho^2}$$

$$\left\{{}_{3\ 4}^{\ 1}\right\}^* = \frac{2mra^3\sin^3\theta\cos\theta}{\rho^4}$$

$$\left\{{}_{4\ 4}^{\ 1}\right\}^* = [r\sin^2\theta - \frac{ma^2\sin^4\theta(r^2-a^2\cos^2\theta)}{\rho^4}]\ \frac{r^2+a^2}{\rho^2}$$

$$\left\{{}_{1\ 1}^{\ 2}\right\}^* = -[\frac{2mr-(r^2+a^2)}{\rho^2}][\frac{m(r^2-a^2\cos^2\theta)}{\rho^4}]$$

$$\left\{{}_{1\ 2}^{\ 2}\right\}^* = \frac{m(r^2-a^2\cos^2\theta)}{\rho^4}$$

$$\left\{{}_{1\ 3}^{\ 2}\right\}^* = 0$$

$$\left\{{}_{1\ 4}^{\ 2}\right\}^* = [\frac{2mr-(r^2+a^2)}{\rho^6}][am\sin^2\theta(r^2-a^2\cos^2\theta)]$$

$$\left\{{}_{2\ 2}^{\ 2}\right\}^* = 0$$

$$\left\{{}_{2\ 3}^{\ 2}\right\}^* = \frac{-a^2\sin\theta\cos\theta}{\rho^2}$$

$$\left\{{}_{2\ 4}^{\ 2}\right\}^* = -a\sin^2\theta\ [\frac{r}{\rho^2} + \frac{m(r^2-a^2\cos^2\theta)}{\rho^4}]$$

$$\left\{{}_{3\ 3}^{\ 2}\right\}^* = r[\frac{2mr-(r^2+a^2)}{\rho^2}]$$

$$\left\{{}_{3\ 4}^{\ 2}\right\}^* = 0$$

$$\left\{{}_{4\ 4}^{\ 2}\right\}^* = [\frac{2mr-(r^2+a^2)}{\rho^2}][r\sin^2\theta - \frac{ma^2\sin^4\theta(r^2-a^2\cos^2\theta)}{\rho^4}]$$

$$\left\{{3 \atop 1\ 1}\right\}^* = \frac{-2mra\cos\theta\sin\theta}{\rho^6}$$

$$\left\{{3 \atop 1\ 2}\right\}^* = 0$$

$$\left\{{3 \atop 1\ 3}\right\}^* = 0$$

$$\left\{{3 \atop 1\ 4}\right\}^* = \frac{2mar(r^2+a^2)\sin\theta\cos\theta}{\rho^6}$$

$$\left\{{3 \atop 2\ 2}\right\}^* = 0$$

$$\left\{{3 \atop 2\ 3}\right\}^* = \frac{r}{\rho^2}$$

$$\left\{{3 \atop 2\ 4}\right\}^* = \frac{-a\sin\theta\cos\theta}{\rho^2}$$

$$\left\{{3 \atop 3\ 3}\right\}^* = \frac{-a^2\sin\theta\cos\theta}{\rho^2}$$

$$\left\{{3 \atop 3\ 4}\right\}^* = 0$$

$$\left\{{3 \atop 4\ 4}\right\}^* = -\frac{1}{\rho^2}\left[(r^2+a^2)\sin\theta\cos\theta + \frac{2mra^2}{\rho^2}\left(1+\frac{(r^2+a^2)\sin^3\theta\cos\theta}{\rho^2}\right)\right]$$

$$\left\{{4 \atop 1\ 1}\right\}^* = \frac{-ma(r^2-a^2\cos^2\theta)}{\rho^6}$$

$$\left\{{4 \atop 1\ 2}\right\}^* = 0$$

$$\left\{{4 \atop 1\ 3}\right\}^* = \frac{-2mar\cot\theta}{\rho^4}$$

$$\left\{ {4 \atop 1\;4} \right\}^* = \frac{a^2 m \sin^2\theta(r^2-a^2\cos^2\theta)}{\rho^6}$$

$$\left\{ {4 \atop 2\;2} \right\}^* = 0$$

$$\left\{ {4 \atop 2\;3} \right\}^* = \frac{a \cot \theta}{\rho^2}$$

$$\left\{ {4 \atop 2\;4} \right\}^* = \frac{r}{\rho^2}$$

$$\left\{ {4 \atop 3\;3} \right\}^* = \frac{ar}{\rho^2}$$

$$\left\{ {4 \atop 3\;4} \right\}^* = \cot \theta + \frac{2mra^2 \sin\theta \cos \theta}{\rho^4}$$

$$\left\{ {4 \atop 4\;4} \right\}^* = \frac{ar \sin^2\theta}{\rho^2} - \frac{ma^3 \sin^4\theta(r^2-a^2\cos^2\theta)}{\rho^6}$$

<u>R_{11}</u>

$$R_{11} = [\log (-g)^{\frac{1}{2}}]_{,11} - [\left\{ {1 \atop 1\;1} \right\}_{,1}+\left\{ {2 \atop 1\;1} \right\}_{,2}+\left\{ {3 \atop 1\;1} \right\}_{,3}+\left\{ {4 \atop 1\;1} \right\}_{,4}]$$

$$+ \left\{ {1 \atop 1\;1} \right\}\left\{ {1 \atop 1\;1} \right\}+\left\{ {1 \atop 2\;1} \right\}\left\{ {2 \atop 1\;1} \right\}+\left\{ {1 \atop 3\;1} \right\}\left\{ {3 \atop 1\;1} \right\}+\left\{ {1 \atop 4\;1} \right\}\left\{ {4 \atop 1\;1} \right\}$$

$$+ \left\{ {2 \atop 1\;1} \right\}\left\{ {1 \atop 2\;1} \right\}+\left\{ {2 \atop 2\;1} \right\}\left\{ {2 \atop 2\;1} \right\}+\left\{ {2 \atop 3\;1} \right\}\left\{ {3 \atop 2\;1} \right\}+\left\{ {2 \atop 4\;1} \right\}\left\{ {4 \atop 2\;1} \right\}$$

$$+ \left\{ {3 \atop 1\;1} \right\}\left\{ {1 \atop 3\;1} \right\}+\left\{ {3 \atop 2\;1} \right\}\left\{ {2 \atop 3\;1} \right\}+\left\{ {3 \atop 3\;1} \right\}\left\{ {3 \atop 3\;1} \right\}+\left\{ {3 \atop 4\;1} \right\}\left\{ {4 \atop 3\;1} \right\}$$

$$+ \left\{ {4 \atop 1\;1} \right\}\left\{ {1 \atop 4\;1} \right\}+\left\{ {4 \atop 2\;1} \right\}\left\{ {2 \atop 4\;1} \right\}+\left\{ {4 \atop 3\;1} \right\}\left\{ {3 \atop 4\;1} \right\}+\left\{ {4 \atop 4\;1} \right\}\left\{ {4 \atop 4\;1} \right\}$$

$$- [(\log (-g)^{\frac{1}{2}})_{,1}\{{}^{1}_{1\,1}\}+\log (-g)^{\frac{1}{2}}_{,2}\{{}^{2}_{1\,1}\}+\log (-g)^{\frac{1}{2}}_{,3}\{{}^{3}_{1\,1}\}$$

$$+ \log(-g)^{\frac{1}{2}}_{,4}\{{}^{4}_{1\,1}\}].$$

The only terms that contribute an m', or m'' are

$$\{{}^{1}_{1\,1}\}\{{}^{1}_{1\,1}\} = \{{}^{1}_{1\,1}\}^*\{{}^{1}_{1\,1}\}^* - \frac{2m'ra^2\sin^2\theta}{\rho^4}\{{}^{1}_{1\,1}\}^* + \frac{m'^2r^2a^4\sin^4\theta}{\rho^8}$$

$$\{{}^{1}_{4\,1}\}\{{}^{4}_{1\,1}\} = \{{}^{1}_{4\,1}\}^*\{{}^{4}_{1\,1}\}^* + \frac{m'ra^3\sin^4\theta}{\rho^4}\{{}^{4}_{1\,1}\}^* - \frac{m'ra}{\rho^4}\{{}^{1}_{1\,4}\}^* - \frac{m'^2r^2a^4\sin^4\theta}{\rho^8}$$

$$\{{}^{4}_{1\,1}\}\{{}^{1}_{4\,1}\} = \{{}^{1}_{1\,4}\}^*\{{}^{4}_{1\,1}\}^* + \frac{m'ra^3\sin^4\theta}{\rho^4}\{{}^{4}_{1\,1}\}^* - \frac{m'ra}{\rho^4}\{{}^{1}_{1\,4}\}^* - \frac{m'^2r^2a^4\sin^4\theta}{\rho^8}$$

$$\{{}^{4}_{4\,1}\}\{{}^{4}_{4\,1}\} = \{{}^{4}_{1\,4}\}^*\{{}^{4}_{4\,1}\}^* + \frac{2m'ra^2\sin^2\theta}{\rho^4}\{{}^{4}_{4\,1}\}^* + \frac{m'^2r^2a^4\sin^4\theta}{\rho^8}$$

$$(\log (-g)^{\frac{1}{2}})_{,2}\{{}^{2}_{1\,1}\} = \frac{2r}{\rho^2}\{{}^{2}_{1\,1}\}$$

$$= \frac{2r}{\rho^2}\{{}^{2}_{1\,1}\}^* + m'\left[\frac{2r^2a^2\sin^2\theta}{\rho^6} - \frac{2r^2}{\rho^4}\right]$$

$$\{{}^{1}_{1\,1}\}_{,1} = \{{}^{1}_{1\,1}\}^*_{,1} - m''\left(\frac{ra^2\sin^2\theta}{\rho^4}\right)$$

$$= \frac{-m'(r^2+a^2)(r^2-a^2\cos^2\theta)}{\rho^6} - m''\left(\frac{ra^2\sin^2\theta}{\rho^4}\right)$$

$$\{{}^{2}_{1\,1}\}_{,2} = \{{}^{2}_{1\,1}\}^*_{,2} + m'\left[\frac{r^2-a^2\cos^2\theta}{\rho^4} + \frac{a^2\sin^2\theta(a^2\cos^2 -3r^2)}{\rho^6}\right]$$

Hence $R_{11} = R^*_{11} + \Gamma_{11}$

$= \Gamma_{11}$

Therefore

$$R_{11} = m'' \left[\frac{ra^2\sin^2\theta}{\rho^4}\right] + m'\left[\frac{1}{\rho^2} + \frac{[2(r^2+a^2)-\rho^2][r^2-a^2\cos^2\theta]}{\rho^6}\right]$$

$\underline{R_{14}}$

$$R_{14} = [\log (-g)^{\frac{1}{2}}]_{,14} - \{{}_{1}{}^{i}{}_{4}\}_{,i} + \{{}_{j}{}^{i}{}_{1}\}\{{}_{i}{}^{j}{}_{4}\}$$

$$- [\log (-g)^{\frac{1}{2}}]_{,i}\{{}_{1}{}^{i}{}_{4}\}$$

the contribution of the m', m'' come from

$$\log (-g)^{\frac{1}{2}}_{,2}\{{}_{1}{}^{2}{}_{4}\} = \log (-g)^{\frac{1}{2}}_{,2}\{{}_{1}{}^{2}{}_{4}\}^* - \frac{m'r^2a^3\sin^4\theta}{\rho^6}$$

$$\{{}_{1}{}^{1}{}_{4}\}_{,1} = m'\left[\frac{a(r^2+a^2)\sin^2\theta(r^2-a^2\cos^2\theta)}{\rho^6}\right] + m''\left[\frac{ra^3\sin^4\theta}{\rho^4}\right]$$

$$\{{}_{1}{}^{2}{}_{4}\}_{,2} = \{{}_{1}{}^{2}{}_{4}\}^*_{,2} - m'\left[\frac{a^3\sin^4\theta}{\rho^4} - \frac{4r^2a^3\sin^4\theta}{\rho^6}\right]$$

$$\{{}_{1}{}^{1}{}_{1}\}\{{}_{1}{}^{1}{}_{4}\} = \{{}_{1}{}^{1}{}_{1}\}^*\{{}_{1}{}^{1}{}_{4}\}^* - \frac{m'ra^2\sin^2\theta}{\rho^4}\{{}_{1}{}^{1}{}_{4}\}^*$$

$$+ \frac{m'ra^3\sin^4\theta}{\rho^4}\{{}_{1}{}^{1}{}_{1}\}^* - \frac{m'^2r^2a^5\sin^6\theta}{\rho^8}$$

$$\{{}^{\;1}_{4\;1}\}\{{}^{\;4}_{1\;4}\} = \{{}^{\;1}_{4\;1}\}^*\{{}^{\;4}_{1\;4}\}^* + m' \frac{ra^3\sin^4\theta}{\rho^4}\{{}^{\;4}_{1\;4}\}^*$$

$$+ m' \frac{ra^2\sin^2\theta}{\rho^4}\{{}^{\;1}_{4\;1}\}^* + m'^2 \frac{r^2a^5\sin^6\theta}{\rho^8}$$

$$\{{}^{\;2}_{1\;1}\}\{{}^{\;1}_{2\;4}\} = \{{}^{\;2}_{1\;1}\}^*\{{}^{\;1}_{2\;4}\}^* + m'[\frac{ra^2\sin^2\theta}{\rho^4} - \frac{r}{\rho^2}]\{{}^{\;1}_{2\;4}\}^*$$

$$\{{}^{\;2}_{4\;1}\}\{{}^{\;4}_{2\;4}\} = \{{}^{\;2}_{4\;1}\}^*\{{}^{\;4}_{2\;4}\}^* - m' \frac{ra^3\sin^4\theta}{\rho^4}\{{}^{\;4}_{2\;4}\}^*$$

$$\{{}^{\;4}_{1\;1}\}\{{}^{\;1}_{4\;4}\} = \{{}^{\;4}_{1\;1}\}^*\{{}^{\;1}_{4\;4}\}^* - m' \frac{ra}{\rho^4}\{{}^{\;1}_{4\;4}\}^*$$

$$- m' \frac{ra^4\sin^6\theta}{\rho^4}\{{}^{\;4}_{1\;1}\}^* + m' \frac{r^2a^5\sin^6\theta}{\rho^8}$$

$$\{{}^{\;4}_{4\;1}\}\{{}^{\;4}_{4\;4}\} = \{{}^{\;4}_{4\;1}\}^*\{{}^{\;4}_{4\;4}\}^* + m' \frac{ra^2\sin^2\theta}{\rho^4}\{{}^{\;4}_{4\;4}\}^*$$

$$- m' \frac{ra^3\sin^4\theta}{\rho^4}\{{}^{\;4}_{1\;4}\}^* - m'^2 \frac{r^2a^5\sin^6\theta}{\rho^8}$$

Hence $R_{14} = m''(-a\sin^2\theta)[\frac{ra^2\sin^2\theta}{\rho^4}]$

$$+ m'(-a\sin^2\theta)[\frac{1}{\rho^2} + \frac{2(r^2+a^2)(r^2-a^2\cos^2\theta)}{\rho^6}]$$

<u>R_{44}</u>

$$R_{44} = [\log(-g)^{\frac{1}{2}}]_{,44} - \{{}^{\;i}_{4\;4}\}_{,i} + \{{}^{\;i}_{j\;4}\}\{{}^{\;j}_{i\;4}\}$$

$$- [\log(-g)^{\frac{1}{2}}]_{,i}\{{}^{\;i}_{4\;4}\} .$$

The terms contributing m' or m'' are

$$\log (-g)^{\frac{1}{2}}_{,2}\{{}^{\;2}_{4\;4}\} = \log (-g)^{\frac{1}{2}}_{,2}\{{}^{\;2}_{4\;4}\}^{*}$$

$$+ m' \left[\frac{2r^2(r^2+a^2)(a^2\sin^4\theta)}{\rho^6}\right]$$

$$\{{}^{\;1}_{4\;4}\}_{,1} = - m' \frac{a^2\sin^4\theta(r^2+a^2)(r^2-a^2\cos^2\theta)}{\rho^6} - m'' \left[\frac{ra^4\sin^6\theta}{\rho^4}\right]$$

$$\{{}^{\;2}_{4\;4}\}_{,2} = \{{}^{\;2}_{4\;4}\}^{*}_{,2} + m'\left[\frac{(3r^2+a^2)(a^2\sin^4\theta)}{\rho^4} - \frac{4r^2(r^2+a^2)a^2\sin^4\theta}{\rho^6}\right]$$

$$\{{}^{\;1}_{1\;4}\}\{{}^{\;1}_{1\;4}\} = \{{}^{\;1}_{1\;4}\}^{*}\{{}^{\;1}_{1\;4}\}^{*} + \frac{2m'ra^3\sin^4\theta}{\rho^4}\{{}^{\;1}_{1\;4}\}^{*} + \frac{m'^2r^2a^6\sin^8\theta}{\rho^8}$$

$$\{{}^{\;1}_{2\;4}\}\{{}^{\;2}_{1\;4}\} = \{{}^{\;1}_{2\;4}\}^{*}\{{}^{\;2}_{1\;4}\}^{*} - \frac{m'ra^3\sin^4\theta}{\rho^4}\{{}^{\;1}_{2\;4}\}^{*}$$

$$\{{}^{\;1}_{4\;4}\}\{{}^{\;4}_{1\;4}\} = \{{}^{\;1}_{4\;4}\}^{*}\{{}^{\;4}_{1\;4}\}^{*} - \frac{m'ra^4\sin^6\theta}{\rho^4}\{{}^{\;4}_{1\;4}\}^{*} + \frac{m'ra^2\sin^2\theta}{\rho^4}\{{}^{\;1}_{4\;4}\}^{*}$$

$$- \frac{m'^2r^2a^6\sin^8\theta}{\rho^8}$$

$$\{{}^{\;2}_{1\;4}\}\{{}^{\;1}_{2\;4}\} = \{{}^{\;2}_{1\;4}\}^{*}\{{}^{\;1}_{2\;4}\}^{*} - \frac{m'ra^3\sin^4\theta}{\rho^4}\{{}^{\;1}_{2\;4}\}^{*}$$

$$\{^{\,2}_{4\,4}\}\{^{\,4}_{2\,4}\} = \{^{\,2}_{4\,4}\}^*\{^{\,4}_{2\,4}\}^* + m' \left[\frac{r(r^2+a^2)(a^2\sin^4\theta)}{\rho^4}\right]\{^{\,4}_{2\,4}\}^*$$

$$\{^{\,4}_{1\,4}\}\{^{\,1}_{4\,4}\} = \{^{\,4}_{1\,4}\}^*\{^{\,1}_{4\,4}\}^* - m' \frac{ra^4\sin^6\theta}{\rho^4}\{^{\,4}_{1\,4}\}^*$$

$$+ m' \frac{ra^2\sin^2\theta}{\rho^4}\{^{\,1}_{4\,4}\}^* - \frac{m'^2r^2a^6\sin^8\theta}{\rho^8}$$

$$\{^{\,4}_{2\,4}\}\{^{\,2}_{4\,4}\} = \{^{\,4}_{2\,4}\}^*\{^{\,2}_{4\,4}\}^* + m'\left[\frac{r(r^2+a^2)(a^2\sin^4\theta)}{\rho^4}\right]\{^{\,4}_{2\,4}\}^*$$

$$\{^{\,4}_{4\,4}\}\{^{\,4}_{4\,4}\} = \{^{\,4}_{4\,4}\}^*\{^{\,4}_{4\,4}\}^* - \frac{2m'ra^3\sin^4\theta}{\rho^4}\{^{\,4}_{4\,4}\}^*$$

$$+ \frac{m'^2r^2a^6\sin^8\theta}{\rho^8} .$$

Hence
$$R_{44} = m''(a^2\sin^4\theta)\left[\frac{ra^2\sin^2\theta}{\rho^4}\right]$$

$$+ m'(a^2\sin^4\theta)\left[\frac{1}{\rho^2} + \frac{[2(r^2+a^2)+\rho^2][r^2-a^2\cos^2\theta]}{\rho^6}\right]$$

<u>R_{12}</u>

$$R_{12} = [\log(-g)^{\frac{1}{2}}]_{,12} - \{^{\,i}_{1\,2}\}_{,i} + \{^{\,i}_{j\,1}\}\{^{\,j}_{i\,2}\}$$

$$- [\log(-g)^{\frac{1}{2}}]_{,i}\{^{\,i}_{1\,2}\}.$$

The terms contributing m' are

$$\{{}^{4}_{1\,1}\}\{{}^{1}_{4\,2}\} = \{{}^{4}_{1\,1}\}^*\{{}^{1}_{4\,2}\}^* - m' \frac{ra}{\rho^4} \{{}^{1}_{4\,2}\}^*$$

$$\{{}^{4}_{4\,1}\}\{{}^{4}_{4\,2}\} = \{{}^{4}_{4\,1}\}^*\{{}^{4}_{4\,2}\}^* + m' \frac{ra^2\sin^2\theta}{\rho^4} \{{}^{4}_{4\,2}\}^*$$

Hence $R_{12} = 0$

$\underline{R_{13}}$

$$R_{13} = [\log\,(-g)^{\frac{1}{2}}]_{,13} - \{{}^{i}_{1\,3}\}_{,i} + \{{}^{i}_{j\,1}\}\{{}^{j}_{i\,3}\}$$

$$- [\log\,(-g)^{\frac{1}{2}}]_{,i}\{{}^{i}_{1\,3}\}$$

The terms contributing m' are

$$\{{}^{1}_{1\,3}\}_{,1} = -m' \frac{2a^2r\sin\theta\cos\theta}{\rho^4}$$

$$\{{}^{1}_{1\,1}\}\{{}^{1}_{1\,3}\} = \{{}^{1}_{1\,1}\}^*\{{}^{1}_{1\,3}\}^* - \{{}^{1}_{1\,3}\}^* m' \frac{ra^2\sin^2\theta}{\rho^4}$$

$$\{{}^{1}_{4\,1}\}\{{}^{4}_{1\,3}\} = \{{}^{1}_{4\,1}\}^*\{{}^{4}_{1\,3}\}^* + \{{}^{4}_{1\,3}\}^* m' \frac{ra^3\sin^4\theta}{\rho^4}$$

$$\{{}_{1}{}^{2}{}_{1}\}\{{}_{2}{}^{1}{}_{3}\} = \{{}_{1}{}^{2}{}_{1}\}^*\{{}_{2}{}^{1}{}_{3}\}^* + \{{}_{2}{}^{1}{}_{3}\}^* \, m' \, \frac{ra^2\sin^2\theta}{\rho^4} - \{{}_{2}{}^{1}{}_{3}\}^* \, \frac{m'r}{\rho^2}$$

$$\{{}_{4}{}^{2}{}_{1}\}\{{}_{2}{}^{4}{}_{3}\} = \{{}_{4}{}^{2}{}_{1}\}^*\{{}_{2}{}^{4}{}_{3}\}^* - \{{}_{2}{}^{4}{}_{3}\}^* \, m' \, \frac{ra^3\sin^4\theta}{\rho^4}$$

$$\{{}_{1}{}^{4}{}_{1}\}\{{}_{4}{}^{1}{}_{3}\} = \{{}_{1}{}^{4}{}_{1}\}^*\{{}_{4}{}^{1}{}_{3}\}^* - \{{}_{4}{}^{1}{}_{3}\}^* \, m' \, \frac{ra}{\rho^4}$$

$$\{{}_{4}{}^{4}{}_{1}\}\{{}_{4}{}^{4}{}_{3}\} = \{{}_{4}{}^{4}{}_{1}\}^*\{{}_{4}{}^{4}{}_{3}\}^* + \{{}_{4}{}^{4}{}_{3}\}^* \, m' \, \frac{ra^2\sin^2\theta}{\rho^4} \, .$$

Hence $$R_{13} = \frac{2m'ra^2\sin\theta\cos\theta}{\rho^4} \, .$$

$\underline{R_{22}}$

$$R_{22} = [\log(-g)^{\frac{1}{2}}]_{,22} - \{{}_{2}{}^{i}{}_{2}\}_{,i} + \{{}_{j}{}^{i}{}_{2}\}\{{}_{i}{}^{j}{}_{2}\}$$

$$- [\log(-g)^{\frac{1}{2}}]_{,i}\{{}_{2}{}^{i}{}_{2}\} \, .$$

There are no contributions of m', m''. Hence $R_{22} = 0$.

$\underline{R_{32}}$

$$R_{32} = [\log(-g)^{\frac{1}{2}}]_{,32} - \{{}_{3}{}^{i}{}_{2}\}_{,i} + \{{}_{j}{}^{i}{}_{3}\}\{{}_{i}{}^{j}{}_{2}\} - [\log(-g)^{\frac{1}{2}}]_{,i}\{{}_{3}{}^{i}{}_{2}\}$$

As in R_{22}, no terms with m' or m'' appear. Hence

$$R_{32} = 0.$$

<u>R_{33}</u>

$$R_{33} = [\log (-g)^{\frac{1}{2}}]_{,33} - \{^{\;i}_{3\;3}\}_{,i} + \{^{\;i}_{j\;3}\}\{^{\;j}_{i\;3}\}$$

$$- [\log (-g)^{\frac{1}{2}}]_{,i}\{^{\;i}_{3\;3}\}$$

No terms with m' or m'' appear. Hence

$$R_{32} = 0.$$

<u>R_{24}</u>

$$R_{24} = [\log (-g)^{\frac{1}{2}}]_{,24} - \{^{\;i}_{2\;4}\}_{,i} + \{^{\;i}_{j\;2}\}\{^{\;j}_{i\;4}\}$$

$$- [\log (-g)^{\frac{1}{2}}]_{,i}\{^{\;i}_{2\;4}\} .$$

The terms contributing m' or m'' are

$$\{^{\;1}_{4\;2}\}\{^{\;4}_{1\;4}\} = \{^{\;1}_{4\;2}\}^{*}\{^{\;4}_{1\;4}\}^{*} + \frac{m'ra^{2}\sin^{2}\theta}{\rho^{4}}\{^{\;1}_{4\;2}\}^{*}$$

$$\{^{\;4}_{4\;2}\}\{^{\;4}_{4\;4}\} = \{^{\;4}_{4\;2}\}^{*}\{^{\;4}_{4\;4}\}^{*} - \frac{m'ra^{3}\sin^{4}\theta}{\rho^{4}}\{^{\;4}_{4\;2}\}^{*} .$$

Hence $$R_{24} = 0.$$

$\underline{R_{34}}$

$$R_{34} = [\log\,(-g)^{\frac{1}{2}}]_{,34} - \{{}_{3}{}^{i}{}_{4}\}_{,i} + \{{}_{j}{}^{i}{}_{3}\}\{{}_{i}{}^{j}{}_{4}\}$$

$$- [\log\,(-g)^{\frac{1}{2}}]_{,i}\{{}_{3}{}^{i}{}_{4}\}\ .$$

The terms containing m' or m'' are

$$\{{}_{3}{}^{1}{}_{4}\}_{,1} = m'\ \frac{2ra^3\sin^3\theta\ \cos\theta}{\rho^4}$$

$$\{{}_{1}{}^{1}{}_{3}\}\{{}_{1}{}^{1}{}_{4}\} = \{{}_{1}{}^{1}{}_{3}\}^*\{{}_{1}{}^{1}{}_{4}\}^* + m'\ \frac{ra^3\sin^4\theta}{\rho^4}\ \{{}_{1}{}^{1}{}_{3}\}^*$$

$$\{{}_{2}{}^{1}{}_{3}\}\{{}_{1}{}^{2}{}_{4}\} = \{{}_{2}{}^{1}{}_{3}\}^*\{{}_{1}{}^{2}{}_{4}\}^* - m'\ \frac{ra^3\sin^4\theta}{\rho^4}\ \{{}_{2}{}^{1}{}_{3}\}^*$$

$$\{{}_{4}{}^{1}{}_{3}\}\{{}_{1}{}^{4}{}_{4}\} = \{{}_{4}{}^{1}{}_{3}\}^*\{{}_{1}{}^{4}{}_{4}\}^* + m'\ \frac{ra^2\sin^2\theta}{\rho^4}\ \{{}_{4}{}^{1}{}_{3}\}^*$$

$$\{{}_{1}{}^{4}{}_{3}\}\{{}_{4}{}^{1}{}_{4}\} = \{{}_{1}{}^{4}{}_{3}\}^*\{{}_{4}{}^{1}{}_{4}\}^* - m'\ \frac{ra^4\sin^6\theta}{\rho^4}\ \{{}_{1}{}^{4}{}_{3}\}^*$$

$$\left\{{4 \atop 2\;3}\right\}\left\{{2 \atop 4\;4}\right\} = \left\{{4 \atop 2\;3}\right\}^*\left\{{2 \atop 4\;4}\right\}^* + m'\left[\frac{r(r^2+a^2)a^2\sin^4\theta}{\rho^4}\right]\left\{{4 \atop 2\;3}\right\}^*$$

$$\left\{{4 \atop 4\;3}\right\}\left\{{4 \atop 4\;4}\right\} = \left\{{4 \atop 3\;4}\right\}^*\left\{{4 \atop 4\;4}\right\}^* - \left\{{4 \atop 3\;4}\right\}^* \frac{m'ra^3\sin^4\theta}{\rho^4}$$

Hence

$$R_{34} = \frac{-2m'ra^3\sin^3\theta\ \cos\theta}{\rho^4}$$

and this completes the computations.

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